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Geometry and Physics Today

Anastasios Mallios[1]

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"*Geometry*," in the sense of the classical differential geometry of smooth manifolds (CDG), is put under scrutiny from the point of view of *Abstract Differential Geometry* (ADG). We explore potential physical implications of viewing things under the light of ADG, especially matters concerning the "*gauge theories*" of modern physics, when the latter are viewed (as they are actually regarded currently) as "*physical theories of a geometrical character*." Thence, "*physical geometry*," in connection with *physical laws* and the associated with them, within the background spacetime manifoldless context of ADG, *"differential" equations*, are also being discussed.

KEY WORDS: general relativity; quantum field theory; singularities; gauge theories; abstract differential geometry (ADG).

"ἀεί ὁ θεός γεωμετρεῖ"
(:*"eternally the God geometrizes"*)

1. By looking at the previous famous utterance (attributed to Plato according to Plutarch—*cf.*, e.g., Smith (1958, p. 88, ft. 4)), as in the frontispiece above, by also taking into account our nowadays perception of Physics, we can say that;

(1.1) *"physical geometry"* is the *outcome of* the *physical laws*.

In this regard, one might also refer here for instance to M. Faraday, as he is quoted by Weyl (1952, p. 169), holding that (emphasis below is ours):

[1] Algebra and Geometry Section, Department of Mathematics, University of Athens, Panepistimioupolis 157 84, Athens, Greece; e-mail: amallios@math.uoa.gr.

(1.2) "... not the field should derive its meaning through its association with matter, but, conversely, ... *particles of matter are ... singularities of the field*."

Now, by looking at the technical correspondence/association,

(1.3) *physical law* $\longleftrightarrow$ $\mathcal{A}$*-connection*,

one realizes that (1.1) might also be construed, as an *equivalent analogue of* the implication;

(1.4) $\mathcal{A}$*-connection* (:*physical law*) $\Longrightarrow$ *curvature*
(: *"geometry,"* alias, *"shaping"*).

Thus, to repeat (1.1) in a different manner, one can maintain that;

(1.5) it is actually the *physical laws* that *make* what we might call, or theoretically construe as, (physical) *"geometry."*

Let it be noted here that in the terminology above, we take for granted the meaning of the technical term *"*$\mathcal{A}$*-connection,"* about which we refer for instance to Mallios (1998), or even to Mallios (1998b, 2005).

Now, it is worthwhile to comment further here on the inverted commas put over the word,

(1.6) *geometry*.

"Geometry" is a composite Greek word (:$\Gamma\epsilon\omega - \mu\epsilon\tau\rho\acute{\iota}\alpha$), the second component being the verb *"metrō"* (:measure) indicating a technical and fundamental fact: *our close entanglement* in our measuring activities with the subject matter that we measure (here, $\Gamma\alpha\acute{\iota}\alpha$: Gk. for *Earth*). In this semantic light, any time the notion of "geometry" is being referred to or used, *by definition* (viz. by the very semantic essence of the word),

(1.7) it is not supposed to actually correspond to something physical (:*real*), but simply to *a model* of ours—one *pertaining to* the description of *reality* (whatever sense we give to the latter concept).

In this line of thought, it is appropriate to recall A.Einstein's maxim, that;

(1.8) *"Time and space are modes by which we think,*
not conditions in which we live."

See for instance Manin (1981, p. 71), as well as, in the same vein, (4.29) in the sequel. [Emphasis in (1.8) above is ours, as it will also be occasionally the case with various quotations in the sequel]. Still, we should mention here the relevant remarks of Bergmann (1979, p. 33), that,

(1.9) "Einstein ... *did not consider geometrization of physics* a foremost
or even *a meaningful objective...*"

(I am indebted here to I. Raptis for bringing to my attention Bergmann's words above). Yet, Bergmann, in the same context of Einstein's unitary field theory (loc. cit.), also insists that what is *of importance is*

(1.10) "*... not a geometric formulation or picturization but a ... fusing of*
the *mathematical structures intended to represent physical fields*."

We remark here that the remarks above are still in accord with (1.1) or (1.5) in the preceding. Thus, we are led again here to propound a

(1.11) *"relational aspect"* of what one might call *"physical geometry."*

In other words, we thus arrive at *something* which is *closer to what we meant by* (1.5). Furthermore, this same aspect is also *akin to what we actually understand*, as we shall also see later on, when we are talking about

(1.12) *"geometry,"* determined *by "differential" equations*—still more, one that corresponds to the *"solution space"* of the latter. The same idea might be regarded as the *source*(!) of the *"cartesian point of view"*; however, *see (1.14)* in the sequel regarding our stance against that perspective within the present *abstract* (thus, *space-independent* (!)) *setting*.

So, within the aforesaid context (see also e.g. (1.11)), we can further maintain that;

(1.13) *"geometrization" of physics* effectively means *"arithmetization" of physics*, for our *"geometry"* is, in effect, *"arithmetical,"* that is, *"cartesian"* (!) in character; hence *not* a *physical* (:natural) one!

Consequently, one comes to realize that

(1.14) *the* previous *association* becomes thus *more natural*, to the extent that it is of a *more "relational"* (!) nature.

However, what is also here of particular physical significance concerning the whole subject matter of the present work, the preceding point of view, as expressed in (1.12), is actually meant in an

(1.15) entirely *"space independent"* way, that is, *not* in a *"cartesian-type"* of manner.

This will become clearer, along with the terminology applied herewith, via the subsequent discussion. Otherwise put, based on the *abstract formalism* of the technique of *Abstract Differential Geometry* (ADG), one is able to

(1.16) formulate *"differential" equations without* resorting to any *background* (*"cartesian"–"newtonian,"* so to say) *"space(time)."*

This, as we shall see in the sequel, is of paramount significance to various problems that *quantum gravity* currently encounters, when these problems are viewed from the standard perspective, viz. from the one of the classical differential geometry of smooth (:$\mathcal{C}^{\infty}$-)manifolds (CDG).

So, in accordance with (1.11), one arrives at a *"leibnizian,"* so to say, *point of view*. That is, following Leibniz himself,

(1.17) we should find a *"geometrical calculus"* that operates directly on the *"geometrical objects"* without the intervention of coordinates.

We may further remark in this respect that the said "*intervention of coordinates*" has been viewed in the past as

(1.18) *"... an act of violence."*

See thus Weyl (1949, 90). At the same time, concerning (1.17), cf., for instance, Bourbaki (1970, Chapt. I; p. 161, ft. 1). Furthermore, within the same context, one has here the relevant remarks of B. Riemann;

(1.19) "Specifications of mass [:measurements] require an *independence of quantity from position*, which can happen in more than one way."

Cf., for example, Mallios (2004, (1.3)). Thus, *in toto*, the preceding sustain the viewpoint that:

(1.20) the description of the *physical laws* (something that could also apply to the *quantum régime*) should be made in such a manner that *no supporting space* or *background scaffolding* (:framework) essentially contributing to that description, *is to be included in our "calculations"* (:rationale); hence, *the latter have* thus *to be* entirely *independent of any notion of "space"* of the aforesaid type.

Now, the aspect of *"description of physical laws,"* as we saw previously in (1.20), can be conceived as just referring to the very *"geometrical calculus"* envisioned by Leibniz (cf. (1.17)), hence to the same *"geometry"* (!) in Leibniz's sense of the word—*ie*, to viewing geometry from a *"relational point of view,"* according to (1.11). Furthermore, the same perspective on

(1.21) *"geometry,"* as *"description*(study) *of physical laws,"*

leads straightforwardly to the custom of

(1.22) doing *"geometry" via "differential" equations* (a fact that actually goes back to René Descartes himself: *"Analytic Geometry"*),

as already hinted at by (1.12). We are going to comment further on the latter aspect within the present abstract setup of ADG in the next Section, also clarifying further our previous remarks in (1.16) above.

2. "Differential" Equations in the Setting of ADG. Functoriality.– As already mentioned above, our aim in the following discussion is essentially to *clarify* (1.16), *and* to dwell further on its *consequences*:

Thus to begin with, we can emphatically remark that, thus far, one of the most effective methods of describing *physical laws* has been the one employing *"differential equations."* In turn, the latter constitute *the* pillar *application of* the (classical) *differential geometry* (CDG), or *Calculus*(!)—still more, of *"the glittering trappings of Analysis,"* to recall here G. D.Birkhoff's expression (see for instance Weinstein (1981, p. 1, ft.2]).

However, the latter (viz. the *classical*, CDG-theoretic) way of describing physical laws, contains in itself the seeds of its own faults (:pathologies), simply by virtue of our previous remarks, as in (1.20), regarding the background "space" (:manifold) employed by CDG. Indeed, by the very essence of *the classical theory* (:CDG), *its whole machinery* (*mechanism*), *is* vitally *rooted on the supporting space* (viz. on the *"locally euclidean"* smooth manifold). Accordingly, as a result of (1.20), one concludes that,

(2.1) *the notion of a* (locally euclidean–smooth–) *manifold* proves thus *not* to be the *appropriate* one for *describing physical laws* (:*"physical reality"*), insofar as the latter refers *to the quantum deep*.

In this context, we may still recall the relevant comments of A. Einstein himself, pertaining to the

(2.2) *inappropriateness of the manifold concept for physical reality*(!).

See for instance Mallios (2006b, (1.6)). At the same time, one can further maintain that,

the aforementioned *shortcoming of the notion of smooth manifold in* various problems of describing physics in *the quantum domain* is mainly *due, not only* (!) *to the way we consider* the *"differential-geometric" mechanism as arising* within the context of CDG (see (3.3) in the sequel), but

(2.3)

(2.3.1) *much more*, because *we* insist on *keeping as a "working framework" the whole "space"*—viz. the entire smooth manifold background arena; and what's more,

by regarding the latter, even *locally*, as the domain of definition of what *we* define as *"differentiable functions."*

Now, the point mentioned in (2.3) above, proves to be, by the concrete working examples presented below, quite an *unnatural way* of trying to apply the *"differential geometric mechanism"* of CDG, since the latter's *character* is entirely *algebraic*(!), as we are going to clarify further in the sequel. Moreover, the same situation occurs when we are confronted with an *extremely pestilential* *anomaly* of the classical theory, in particular when we insist on applying the latter *to the quantum deep*. Indeed, this anomaly usually manifests itself in the guise *of "infinities"* (:*"singularities"*)! However, all these anomalies, be as they may from the CDG-theoretic vantage, *are not* actually *real* (!) (cf. the aforementioned examples, as will be presented in what follows).

On the other hand, we shall further illuminate *the situation that appears within the quantum framework*, when looking at it *from the point of view of the abstract theory*, by briefly summarizing the relevant conclusions into the following.

Scholium 2.1.– When looking at the fundamentals of *quantum theory*, in conjunction with potential applications in that context of (differential) geometry, one actually realizes that;

(2.4) we usually *associate numbers* (à la Descartes) *to a space that*, in effect, *does not exist*(!). And it does not, at least in the sense that *we* ascribe to it our own *"spatial perspective"* which is, in point of fact, *always cartesian*(!). Of course, this subjective perspective turns out *not to be in accord with* our (experimental) knowledge about physics in *the quantum régime*.

Thus, we are indeed trapped here by our own perspective, our own assumption of a background *"locally euclidean,"* viz. *"newtonian,"* in character "space", via which we look at and actually exercise (:implement) our *"differential calculus."* This, plainly, results in the pseudo-physical correspondence manifold $\leftrightarrow$ spacetime, which has stymied our theories of the quantum domain in the form of the aforesaid anomalies. *In summa*, we are

(2.5) *unable to apply* the classical (:newtonian) aspect of *differential geometry in the "quantum deep,"* due mainly to the emergence of the so-called *"singularities,"* and other relevant anomalies. [As already noted several times in the preceding, the latter phenomenon is actually due to the *particular type of* our (*"smooth"*) *functions involved*, that *"smoothness" being* in turn *a direct reflection* of the sort *of "space"* (:locally euclidean) *we use*!].

Consequently, *to stress it once more*,

from the "leibnizian" (:relational) vantage of ADG, it is not the functions we use (viz., when regarded as the carriers of the *"differential geometric-mechanism"* which are *inappropriate concerning physical descriptions in the quantum deep, but*, simply, *the "space"* on which *the* said *functions are supposed to be defined*. Such *a space*, as that one we try to apply (viz. the "locally euclidean" one), *is* physically *non-existent*(!); it is *we* that insist on trying to formulate our physical descriptions (:theories) of the quantum domain by *a priori* forcing it into them.

(2.6) In this and in a wider context, given that, an *"arithmetical space,"* as it actually is the standard *"euclidean/ cartesian space"* which we traditionally employ in the *classical theory* (:CDG), is not *"physical"* (:*real*)(!) in view of the troubles (:singularities, infinities, and other anomalies) we get when we apply it to our attempted descriptions of the *"quantum deep."*

[We thus get, in this respect, even an *"experimental"* (:concrete) *manifestation* of the *ineffectiveness of* our (spatial) *model*!].

That is, the *"space"* model we usually assume in our physical theories—which we seem to persistently identify with *"physical space"*—is entirely a *numerical one*." In this identification, we are largely influenced (or even, biased!) by the formidable successes the manifold model has so far enjoyed in our "macroscopic" theories. This model, however, proves inadequate, and at times it "collapses," when confronted with physics in the quantum deep.

So, in other words, we are thus entrapped by the *particular (experimental) successes* that the aforesaid point of view (:the classical one) has enjoyed in the past, namely, those concerning our experience/applications *"in the large"* (:macroscopically).

Now, within the same vein of ideas, and still in connection with the essentially (*categorical*) *correspondence* between *"space"* and *functions*, one actually has the following *"identification,"*

$$\text{(2.8)} \qquad \textit{functions} \longleftrightarrow \textit{space, ``Gel'fand duality''}$$

which we may also call (as noted above), *"Gel'fand duality,"* a fact explicitly pointed out, in its full generality, by the language of the *theory of* (especially, *non-normed*) *Topological Algebras* (see, for instance, A. Mallios [TA; p. 223, Theorem 1.2, as well as, p. 227, Theorem 2.1]).

Now, as already hinted at in the foregoing, something which will also be considered in the ensuing discussion, the previous situation has effectively nothing to do with the *mechanism itself* of the aforesaid classical theory (:differential geometry), with the latter machinery being essentially *"leibnizian"* (!) in nature. Precisely this has been pointed out time and again by what we may call *"Abstract Differential Geometry"* (:ADG); see thus Mallios (1998b), as well as Mallios (2005).

In toto, the preceding represent the way one may look at what we usually understand nowadays by the term *"space"* (speaking, of course, within the context of what is called *"mathematical physics"*). The fact is that the previous thoughts are the *outcomes* (the distilled didactics) of our experience with ADG as a general theoretical framework, while the same didactics can be drawn from numerous applications of the theory to problems in "*quantum relativity*" Finkelstein (1997), as the latter has been explained already in other places (see, for instance, Mallios (2005), as well as, Mallios and Raptis (2003), (2005)). So it is this entirely *new* (*axiomatic*) *perspective of* ADG, pertaining to the *inherent mechanism of* the classical *differential geometry* (:CDG), which provides several potential applications to quantum gravity, while the same mechanism may also prove to be in accord with the *"spatial" situation* one is confronted with *physics in the quantum deep*, as already hinted at above; in this regard, see also e.g. Mallios and Rosinger (2001), along with Mallios and Raptis (2005). Further illuminating comments on this last issue are going to be presented through the discussion in the sequel.

On the other hand, by looking at the whole *classical set-up from the point of view of* ADG, we can further point out here that in *complete contrast to the classical case*, (2.9) *the framework of* ADG *does not*, as a matter of principle, *depend on any background "space"* (:carrier, think e.g. of *"space-time"* in the classical domain) which would contribute to its *"differential"* mechanism, the latter being thus *entirely rooted on* $\mathcal{A}$(!)—our *"generalized arithmetics,"* alias, *"sheaf of coefficients."*

Moreover, the issue in (2.6) above constitutes

(2.10) the quintessence of the *quantum field-theoretic character of* ADG.

Indeed, *the* whole *set-up of* ADG avails itself for

(2.11) *formulating* our *equations in* a *quantum field-theoretic manner*, viz. *quantum-relativistically*!

In connection with this, see also our previous relevant remarks in Mallios (2004, (9.8), (9.23), along with Section 11 therein). Yet, to state the point above in an equivalent way;

(2.12) *it is* actually *we who express the* (physical) *laws* as *"differential" equations* by means of our *"arithmetics"*—in our case, through *the* ($\mathbb{C}$-algebra) *sheaf* $\mathcal{A}$. At the same time, the same machinery (:*calculus,"* à la Leibniz, or even *"differential geometry"*) is *based on* $\mathcal{A}$, and not at all on any background *"space"* as in the classical case (:CDG).
Of course, the latter fact is of *paramount importance*, when we are confronted with *quantum gravity* problems.

In the same line of thought, one may quote some remarks of Baez from (1994, beginning of Preface), holding that (emphasis below is ours):

(2.13) "A fundamental problem with *quantum* ... *gravity* ... is that *in* ... *general relativity* there is no background geometry to work with: *the geometry* of spacetime itself *becomes a dynamical variable*."

On the other hand, the aforementioned (see (2.9), (2.12))

(2.14) *independence of* the *"differential" mechanism of* ADG *from any background space*, enables us to regard that mechanism as a *"variable"* entity too—this being the case by its very "construction", since it is entirely *based on* (reduced to) $\mathcal{A}$. Thus, what we also understand as *"differential" geometry* (:*"geometrical calculus,"* à la Leibniz) which goes hand in hand with the said mechanism, can also be regarded as being *"variable."*

Furthermore, something that is also *fundamental* in this respect, the same *"geometrical calculus"* (hence the concomitant *"geometry"* too) becomes simply *"relational,"* as it refers directly to the *"geometrical objects"* (in our case, the *vector sheaves*) themselves, without the interference of any *"space"* in the classical sense of the latter term.

In connection with the above, we may also recall for our convenience that, *technically speaking*, here we suppose that we are thinking in terms of an (*abstract*) *"differential setting,"* based on a given *"differential triad,"*

$$(\mathcal{A}, \partial, \Omega) \tag{2.15}$$

over an (arbitrary, in general) *topological space* X, which in turn serves as the base space of all the sheaves involved throughout the theory. Now, within this context, a *"geometrical object"*—*eg*, an elementary particle—can be associated with what we call a *Yang-Mills field*, viz. a pair

$$(\mathcal{E}, D), \tag{2.16}$$

consisting of a *vector sheaf* $\mathcal{E}$ on X and an $\mathcal{A}$*-connection* D *on* it; see e.g. Mallios (2006b (3.2), (3.3)), or even (Mallios, 2005: Vol. II, Chapt. I). It is actually in terms of such pairs as the one above that *"differential" equations* are set up in the framework of ADG (loc. cit.).

Thus, within the above framework, we can further refer here to a *fundamental principle* underlying thus far the whole ADG-machinery, namely, that;

(2.17) *everything* that *we want to ascribe to* a pair $(\mathcal{E}, D)$, as above, *is* virtually *reduced to* a similar condition/asumption for *the pair* $(\mathcal{A}, \partial)$ (see (2.15)). Occasionally, the said reduction can be effectuated under appropriate, in principle *only*(!) *topological*, hypotheses for X (see also the comments below).

As already noted before, the context of (2.17) exhibits, in point of fact, the *"Leitmotiv"* that actually dominates the very technique of ADG; see thus Mallios [VS], or even Mallios (2005). On the other hand, the same ensures also the

(2.18) *"covariance" of* the whole setting of ADG, *with respect to* $\mathcal{A}$.

Thus, the *"variance"* here is always *relative to our* own *"arithmetics,"* or even (*generalized*) *domain of coefficients,"* yet, *"structure sheaf"* $\mathcal{A}$, which is by assumption a *unital commutative* $\mathbb{C}$*-algebra sheaf* on X; see (2.15). So, strictly speaking it is *we* who *measure*(!)/*calculate*, while, and this is *of* special *significance* as noted earlier, the whole framework/calculations of ADG is effectuated *without leaning upon any background "space"* (:carrier), as for instance the background "space-time"(!) of the classical case.

On the other hand, we can further say that;

(2.19) *physical laws are* always *"functorial", this being the only way we actually perceive them!*.

Of course, in the statement above we essentially *"abuse language,"* as we explain below.

Note 2.1.– Looking at the sense in which we actually use the term

"functorial"

as in (2.19) above, and also taking into consideration (1.2) earlier, it should be pointed out here that;

(2.20) *the aforesaid term is always meant with respect to us*(!), viz., relative to (our *"generalized arithmetics"*) $\mathcal{A}$.

Therefore, what we actually consider in (2.19) is the *manifestation of the physical laws*! Indeed, their description, expression, hence also their potential application (: effectuation) via $\mathcal{A}$!

Now, this goes hand in hand with (or even, it is something that is in point of fact an *equivalent expression* of) the *"principle of general covariance"* of general relativity. Accordingly, by considering, as we did it in the preceding, *differential equations as expressing physical laws* (see (1.5), (1.12)), we realize that, indeed,

(2.21) *differential equations* should be, by their very definitions, *"functorial" in nature*! Consequently, their formulation should be made *in terms of "functorial objects."*

Now, by the last term, technically speaking, we mean something that by definition is $\mathcal{A}$-*invariant*; alias, a *"tensor,"* in the sense that it respects our *"arithmetics"* $\mathcal{A}$.

Furthermore, (2.19) can still be construed, as a corollary of (1.2), together with (1.5) in the preceding. Thus, by further considering (see also (2.20) above) the

physical laws, as *the manifestation of* the (deepest physical) *dynamics* (:*"causality"*), one comes to the conclusion that;

(2.22)

(2.22.1) *"dynamics"* should be *"functorial,"* as well,

whenever *we* actually effectuate it (viz. the physical law, cf. also (1.4)). Therefore, this very realization or representation of it (*by us*(!), of course) becomes *"functorial,"* or even *"tensorial"* too; hence, the same physis of the *curvature* (:*"geometry"*), see also (1.4), as before.

Now, further commenting on our last conclusion above, we may still recall that, according to our axiomatics,

(2.23)

the *curvature* (:field strength) is the manifestation (effectuation, realization or representation) of the *"identification"* (correspondence, cf. also (1.4)),

(2.23.1) *dynamics* (:"causality") $\longleftrightarrow$ ($\mathcal{A}$-)*connection*, therefore (see also (2.22.1)), the *tensorial* (*functorial*, cf. (2.20)) *aspect of* the *curvature*.

In connection with the above, we can still note that the aforementioned *functorial/tensorial character of the curvature* in the sense of (2.20), which is always *the outcome* (field strength) *of a* given *"field"* (:$\mathcal{A}$-connection, see, for instance, Mallios (2006b, (3.15)) or even (Mallios (2004), (3.21.1))), *is* further *expressed* by the familiar relation,

(2.24) $$\nabla\rho = 0,$$

yet, *equivalently* (in ADG-theoretic terms), by the relation;

(2.25) $$D_{\mathcal{H}om(\mathcal{E},\mathcal{E}^*)}(\tilde{\rho}) = 0,$$

where we still have;

(2.26) $$\mathcal{H}om(\mathcal{E},\mathcal{E}^*) = \mathcal{E}^* \otimes_{\mathcal{A}} \mathcal{E}^* = (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})^*.$$

See A. Mallios [VS: Chapt. VII; p. 165, (8.70), along with Chapt. IV; p. 302, Theorem 6.1 and p. 305: (6.16)]; thus, we have herewith the so-called, classically, *"Levi-Civita identity."* By further referring to the above notation, we consider therein a given *Yang-Mills field*

(2.27) $$(\mathcal{E}, D),$$

see loc. cit., Chapt. IX; p. 244, along with (2.15), as above, while ρ stands there for a *Riemannian $\mathcal{A}$-metric* on $\mathcal{E}$, *"compatible with D"* (ibid., Chapt. VII; Section 8). It is worth noticing here that the previous condition on the pair

(2.28) $$(D, \rho),$$

as above, is actually the *upshot of a similar assumption for the* standard *pair*,

(2.29) $$(\mathcal{A}, \partial),$$

cf. (2.14), under appropriate supplementary conditions on the items involved herewith, these being in the case of X, *only topological* ones (cf. thus (2.16) in the preceding); yet, in that context, see also A. Mallios [VS: Chapt. VII; p. 168, Theorem 9.1: *Fundamental lemma of Riemannian vector sheaves*]. Accordingly, we further understand here that (: the *"physical significance" of* (2.25)),

(2.30) *to "realize" the curvature*, one has to *"compare" it with something else*!

We terminate the present Section with the following remarks of N. Bohr, as quoted by Auyang (1995, p. 229)), referring to the way one actually has to "look at" the Nature; indeed, with the same remarks the foregoing rationale and related remarks thereon are really in accord, as it actually concerns *our relevance*, with respect to the observed physical laws, which, technically speaking, as it was pointed out in the preceding, is *expressed*, in effect, *through the "structure sheaf"* $\mathcal{A}$*, independently of any surrounding*/supporting *"space."* Thus, according to the aforementioned remarks (emphasis below is ours),

(2.31) "It is wrong to think that the task of physics is to point out *how nature is*. *Physics concerns what we can say about nature*."

Consequently, to follow in that context the favorite expression of A. Einstein himself, *we* thus always *"describe," not explain* (!), the *physical applications* of every day life being, therefore, simply *consequences of the former* (descriptions), as above(!). Yet, within the same vein ideas, we may still quote, herewith, Wittgenstein (2003, p. 17), in that;

(2.32) *"Physics does not explain anything; it simply describes concomitant cases."*

(Emphasis above is ours). Therefore, as already emphasized in the preceding, we do not actually explain *"anything,"* through Physics as far as the *physical laws* (:physis) are concerned, but we just *describe*/study *their consequences*(!) In spite of the latter function, it undoubtedly appears (see *applications*) that,

(2.33) *we do understand, to a certain extent*(!), *the way* that *these laws work*!

3. ADG as a Scheme Applicable in the Quantum Deep.– Our purpose, by the ensuing discussion, as the title of this Section indicates, is to further clarify the way one can look at a *potential application of* ADG *in the quantum régime*, thus, in point of fact, of *the very mechanism of the classical* (:*"newtonian"*) *differential geometry*, very effective(!), for that matter, so far, however, *now, within the aforesaid domain, but*, already *from the point of view of* ADG (viz. *axiomatically*), thus, *freed from* its *"beautiful shackles"* (C.J.Isham); indeed, it is proved that the latter *obstacles* are *due*, simply, *to the entanglement*, according to the classical theory, *of the* same *mechanism with the "locally euclidean" nature* of that theory (in effect, *much more*, because of the *maintenance of the* whole *"smooth setting", as* a *working framework*, in this context, cf. also (2.3) in the preceding), the latter being also considered, in view of the same standard theory (CDG), *the only source*(!), within that context, *of* the all powerful (infinitesimal/integral) *Calculus*, hence, of the classical differential-geometric machinery, as well. So, it is here exactly that a *supreme didagma of* ADG comes just to the foreground, *in fact*;

(3.1) *the differential-geometric mechanism of* the *classical differential geometry* (CDG)–being, in effect, *of* a *leibnizian character–can*, equally well, *be supplied, by other sources, apart from a "locally euclidean" space*/(smooth) manifold, its *existence* being thus *independent of any* such *"space."*

Furthermore, as already pointed out in the preceding (see, for instance, the quoted citations, in that context, of Einstein, Feynman, Isham), a *"space,"* as *in* the *latter part of (3.1)*, together *with its "differential set-up," is* entirely *out of the question for the quantum deep*(!).

On the other hand, by further commenting, within the preceding vein of ideas, on the basis of our experience from ADG, as exposed above, we realize that one can virtually interrelate well-known phenomena in the past with still existing tendencies in quantum physics of today:

Thus, the heuristic *opposition of Einstein against Quantum Field Theory* (:*"the other Einstein,"* see e.g. Stachel (1993, p. 283, 285)) might also be viewed, apart from other physical reasons, still, as an outcome of the *failure of* classical differential geometry –hence, in particular, of general relativity too– as it concerns *the way the inherent in that theory* (differential) *"Calculus" is supplied, to cope with problems of the quantum theory*. Indeed, we can further say that, looking at the same classical (:*"newtonian*) manner of definition of the *"derivative,"*

(3.2) Einstein was demanding, within that framework, to *abandon, even* the notion of *continuity*(!) *in physics*, having thus, instead, to invent a *"purely algebraic physics"* (loc. cit., p. 285). Moreover, we could add to his imperative in view of the above, we should look for a (purely) *algebraic analysis*(!)

In this connection, we can certainly refer here, as already done in the preceding, to the relevant remarks thereof of Feynman (1992, p. 166), as well as, to those of Isham (1984, p. 393) (in this regard, see also e.g. Mallios (2006b)), concerning the *ineffectiveness of* the *classical differential geometry*, and *in extenso* of that one of a *smooth* (:$\mathcal{C}^\infty$-)*manifold*, *within the quantum régime* (yet, see (2.3) in the foregoing, along with our discussion in the subsequent Section 4).

On the other hand, the *pertinence*, in that context, *of* ADG *to* confronting with problems of *quantum gravity* still lies in its *algebraic* (viz. *"leibnizian,"* so to say) *character*: Indeed, the whole edifice of ADG is, by its very construction, *sheaf-theoretic, sheaf theory* being, of course, of an *algebraic nature* (see, for instance, Grauert and Remmert (1984, p. VII)). Thus, ADG *might* also *be construed*, as an

(3.3) *algebraic* (:*"leibnizian"*) *manner of presenting* the fundamentals of the *classical differential geometry*, while, at the same time, still getting, as an *outstanding outcome* (see, for example, (2.22), as well as, (2.1) in the preceding), the *possibility of working, without any resort to a background "space,"* in the classical sense of the latter term, as for instance, to a *"space-time continuum"* (!), as it happens, instead, in the standard theory.

Certainly, the significance of the aforementioned *two issues of* ADG cannot be underestimated, while the same might be, in effect, quite well, what A. Einstein himself, by 1935 already, was looking for (see, for instance, still, Stachel (1993, p. 285), as above).

4. Particular Potential Applications of ADG in the Quantum Régime.– We start, by presenting, within the framework of ADG, the relevant *theory of Elemér E. Rosinger*, pertaining to *"generalized functions,"* whose *algebra* (*sheaf*), in particular, the *"foamy"* one, can be used, as a *"sheaf of coefficients,"* defining thus, appropriately, a corresponding herewith *"differential triad,"* basic ingredient to having a set-up in developing the mechanism of ADG (see (2.9), (2.20) in the foregoing). For similar previous accounts, see also Mallios and Rosinger (1999, 2001), as well as, Mallios (2005, Vol. II, Chapt. IV; Section 5).

However, before we come to the relevant exposition, it is still to be noticed, herewith, a *fact of a particular significance*, referring to the very *structure of* ADG (cf. (4.1) below, along with Subsection 4.(b) in the sequel), as it concerns two *important special cases* of the general theory of ADG, we are going to consider, by the subsequent discussion; the same are also characteristic of the way, one may have a *"differential-geometric mechanism,"* in the sense of ADG, *different*, in character, *from the classical* manner of obtaining it (viz., via smooth manifolds, *but*, see also (4.1), along with (4.11) below). So it is, indeed quite useful (yet, rather, necessary(!)) to make the following remarks. That is,

(4.1) even, if we take, as the *base space of* the *sheaves* involved, within the *abstract context of* ADG, a (*smooth*) *manifold* X, in the standard sense of this term (cf. thus the ensuing two Subsections below), *its rôle* (:as *the* source of Calculus) *is* actually *transferred* now *to the "sheaf of coefficients,"* $\mathcal{A}$. Yet, this is very *organic*, since it is essentially *we*(!), who *make* the *calculations*/experiments, based on our own *"arithmetics,"* viz. for the case in hand, again, via the *algebra sheaf* $\mathcal{A}$.

Therefore,

(4.2) the manifold X, as in (3.1) above, is just viewed, simply, as a particular *topological space*, being, of course, by its very definition, *paracompact* (*Hausdorff*); the latter condition is certainly, otherwise, very useful, indeed, when referring to *cohomological* issues: *sheaf cohomology* is, for that matter, apart from *sheaf theory* itself, the other fundamental ingredient of ADG. Yet, it may still happen that the *topology of* X be chosen quite *different from* the initial, viz. the *standard* topology of the manifold X, i.e., the *"locally euclidean"* one); see, for instance, *"Sorkin's topology"* in Subsection 4.(b) below.

However, as we shall see, by the ensuing discussion, the particular cases we look at in the sequel, *do have*, so to say, *a*

$$\textit{newtonian spark}\,(!), \tag{4.3}$$

that is, something of a *"starting point,"* that will become better clear, by the subsequent rationale. Notwithstanding, as we shall also realize, in that context, (4.4) *this does not affect*, at all(!), *the "leibnizian"* character of the *mechanism of* ADG,

as the latter is *inherently afforded, by the* same two particular *subsequent examples* of the general theory.

The preceding certainly constitutes a *fundamental* special *issue* of *paramount importance*, indeed, for potential applications; the same could still be worthwhile to be viewed *axiomatically*(!), as well, contributing thus to our knowledge, as it concerns the *whole character of the general theory*. In this regard, see also our previous account thereof, already in Mallios (2005, Vol. II, Chapter IV; Section 5).

> **Note 4.1.**– By still referring to our previous issue in (4.3), as we shall see, by the ensuing examples the so-called therein *"newtonian spark"* not only supplies the *"structure sheaf"* $\mathcal{A}$, by the *"spark"* (fuse) of its *"differential" mechanism*, but what is, in effect, herewith of a particular importance, is that *one assures*, in that context, the validity of *Poincaré Lemma*, indeed, of an *extraordinary importance* of the whole mechanism of ADG. Thus, one can complete (4.3), by actually setting the *equivalence*;

(4.5) *"newtonian spark"* $\Longleftrightarrow$ *Poincaré Lemma*.

So here again one realizes the *fitness of*

(4.6) *replacing of the "geometric character," locally*(!), *of classical analysis, by cohomological issues*.

However, more on this we shall see in the pertinent places below.

Thus, we come now to examine our first Example, pertaining to the situation described by (4.3), (4.5) above, straightforwardly, by the ensuing Subsection:

4. (a). Rosinger's Algebra Sheaf.– Here the aforementioned already *"newtonian spark,"* as in (4.3) above, is nothing more, as we shall presently see, right below, than *the classical*

(4.7) *"dx"*

of the *standard theory of* $\mathcal{C}^\infty$*-manifolds*. Thus, *the* above *classical* "dx" is, for the case at issue, prolonged, true, *it is*, in point of fact, *"promoted"* (!), so to speak, *to* an abstract,

(4.8) $$\text{“}\partial\text{”}$$

in the sense of ADG (see, for instance, (2.14) in the preceding), defined now on an *algebra sheaf* (Rosinger's), containing the standard one $\mathcal{C}_X^\infty$, viz. the $\mathbb{C}$-*algebra sheaf* (of germs of $\mathbb{C}$-valued smooth functions [$\mathbb{R}$-valued functions could also be considered, of course]) on a given manifold X. Indeed, we can still say, in anticipation, that *Rosinger's algebra sheaf* $\mathcal{A}_{\mathrm{nd}}$, and, in extenso $\mathcal{A}_{\mathrm{foam}}$ (see (4.20), (4.21) in the sequel) *contain much more than* $\mathcal{C}_X^\infty$ of the classical theory (cf. thus (4.14) below). We depict the above, by the following diagram, whose notation will become more clear, through the ensuing discussion. Thus, *we have*;

(4.9)
$$\begin{tikzcd}
\mathcal{C}_X^\infty \arrow[r, "d"] \arrow[d, "\cap"'] & \Omega_X^1 \arrow[d, "\cap"'] \\
\mathcal{A}_{nd} \arrow[r, "\partial"'] & \Omega_{nd}^1 \equiv \Omega^1
\end{tikzcd}$$

We proceed, by explaining the notation applied in (4.9); thus,

(4.10) $$\mathcal{A}_{nd} \equiv \mathcal{A},$$

stands therein for *Rosinger's algebra sheaf* a $\mathbb{C}$-*algebra sheaf* on X, the latter space being, by assumption, an *open* subset *of* $\mathbb{R}^k$. However, since the whole theory is, in point of fact, of a *local nature*, one may consider, instead, $\mathbb{R}^k$ *just locally*, that is, we can assume that X is a *smooth* (:$\mathcal{C}^\infty$-)*manifold*. Notwithstanding, for simplicity's sake, *we adopt*, throughout, *that*

(4.11) $$X \text{ is open in } \mathbb{R}^k.$$

Accordingly, X *being*, by its very definition, a *metrizable* space, one concludes, in particular, that

(4.12) X is a *paracompact Hausdorff* (topological) *space*.

See, for instance, Dugundji (1966, p. 186, Theorem 5.3).

Now, *Rosinger's algebra sheaf* $\mathcal{A} \equiv \mathcal{A}_{nd}$, as in (4.10) above, is actually an appropriate (cf. (4.13) in the sequel) *quotient of a functional* (algebra) *sheaf*: thus, technically speaking, it is defined, as a *quotient of a functional* (algebra) *presheaf*, the latter being proved, in particular, to be a *"complete"* one, therefore (J. Leray), a *sheaf*. Yet, the corresponding, in that context, *quotient algebras* are defined, modulo a suitable (2-sided) *ideal* (:*"Rosinger's ideal"*), which is essentially *characterized, by* what we may consider, as *"Rosinger's asymptotic vanishing condition"*; in particular, the latter is defined, via a

(4.13) *closed nowhere dense* (hence, the subindex "nd", appeared in (4.10)) *subset* Γ *of* X, the same ideal consisting thus of those *functions/* elemenets of the (local section) algebras concerned, that *vanish "eventually"* (w.r.t. a parameter involved, a natural number, index) *on any relatively compact* subset *of the complement of* Γ.

Concerning the precise definition of the preceding, we refer to Mallios (2005, Vol. II, Chapt. IV; Section 5), or even (:to Mallios and Rosinger (1999, p. 236; (2.2)). Yet, by further looking at the same sheaf (4.10), as above, and also complementing the information we have through (4.9), we still note that we actually get, by the very definition of (4.10) (cf., for instance, Rosinger (1990, p. 8; (1.2.15), (1.2.16), along with p. 367, (2))),

$$\mathcal{C}_X^\infty \subsetneqq \mathfrak{D}'_X \subseteq \mathcal{A} \equiv \mathcal{A}_{nd}. \tag{4.14}$$

Here the middle term in (4.14) denotes the *sheaf* (of germs) *of Schwartz distributions* on X, viewed, as a $\mathbb{C}$-*vector space sheaf* on X (loc. cit., (5.19)).

On the other hand, the *"basic differential operator"*

$$\partial : \mathcal{A} \longrightarrow \Omega^1, \tag{4.15}$$

that one has to define, according to the general theory of ADG, see, for instance, (2.14) in the preceding, or even in Mallios (1998, Chapt. VI; Section 1), is here provided by the presence of the first member in (4.14), that is, *locally*, by that one of a ($\mathbb{C}$-)*algebra* of the form

$$\mathcal{C}^{\infty}(U), \text{ with } U \textit{ open in } X \subseteq \mathbb{R}^k, \tag{4.16}$$

(see also (4.11)), that virtually constitutes, within the present context, the *"newtonian spark,"* hinted at in (4.3). Thus, the *basic differential* ∂, as in (4.15), in now defined *coordinate-wise*, along the classical patterns, since the *basic constituents of* the *Rosinger's algebra* (*pre*)*sheaf* are (local sections of) *cartesian product algebras* of the form,

$$(\mathcal{C}^{\infty}(U))^{\mathbb{N}}, \tag{4.17}$$

with U, as in (4.16), which then are *"quotiented", according to* (4.11). Of course, the previously coordinate-wise (*classically*!) defined *differential* passes to the quotient. For technical details see Mallios (2005, Vol. II, Chapt. IV; Subsection 5.(b)), or even to Mallios and Rosinger (1999). Thus, the overall moral, that is here, concluded according to the *general principles of* ADG, is the following;

(4.18) Starting *from any* basic *"differential triad,"* in the sense of ADG (even a classical one, as e.g. a *"locally euclidean* one, this is the case, herewith, *we* can then *perform any* (functorial) *operation, provided within the category of differential triads*, to get thus at a new one [occasionally, more useful/flexible than the initially given one!].

Thus, by referring, in particular, to *Rosinger's algebra sheaf,* as above, *and the associated* with it *differential triad*, we remark that, in view of (4.18), what we actually consider, in that context, is:

i) to take a *denumerable cartesian product* of the standard (:*newtonian-cartesian*) *differential triad*

$$\left(\mathcal{C}^{\infty}_{X}, d, \Omega^1\right), \tag{4.19}$$

as well as,

ii) to take, in particular, *a* pertinent *quotient* of the above, modulo *Rosinger's ideal*, as indicated by (4.13).

In this connection, the aforesaid *categorical treatment of* ADG, has been occasionally considered already in Mallios (1998, Chapt. VI; Sections 5 and 6), as well as, in (Mallios, 2005, Chapt. I; Section 5.(e), 5.(f): *"pull-back"* functor]; yet, an analogous fuller and systematic *categorical study of differential triads* has been recently supplied by the relevant work of Papatriantafillou (2000, 2004, in preparation).

On the other hand, one gets at an *immense generalization* of the above, by considering, in place of $\mathcal{A}_{nd}$, what we may call a *Rosinger's multi-foam algebra sheaf*, along with the associated *differential triad*,

$$(B_{\Lambda,J}, \partial, \Omega^1); \tag{4.20}$$

here the space X, base of the sheaves concerned, is still given by (4.11), while the sheaf on X appeared in the first member of (4.20) is again a pertinent *quotient* of the ($\mathbb{C}$-)algebra

$$\mathcal{C}^\infty(X)^\Lambda, \tag{4.21}$$

with Λ an *upwards directed set*, modulo an analogously defined (2-sided) *ideal* of the same algebra, with respect to a given (upwards) *directed family* J of *"residual"* subsets of X, the *"singularity-sets"* of X (viz. those $A \subseteq X$, with $\overline{CA} = X$), the applied terminology, herewith, being hinted at potential physical applications: See Mallios and Rosinger (2001), as well as, Mallios (2005, Vol. II, Chapt. IV; Section 6). Of course, the *singularity-sets*, as above, *generalize* the notion of *nowhere dense sets*, considered by (4.13) in the preceding. Hence, the *increase of the types of "singularities," one can cope with*, in the framework of ADG, as explained in the foregoing.

Now, *the* same *moral*, that dominates our previous comments *in* (4.18), is, in point of fact, as we shall presently see in the sequel, the *prevalent point of view* also *in* the ensuing example, referring to another *potential application* of the very technique of ADG *in* problems of *quantum gravity*.

4.(b). Finitary Incidence Algebra Sheaves.—Similarly to the preceding Example 4.(a), here too, as already said, for that matter, one starts again from a *smooth* (:$\mathcal{C}^\infty$-)*manifold* X, that still, for simplicity's sake, we assume that it is just an *open* subset *of* the *euclidean space* $\mathbb{R}^k$ (see (4.11)). However, as we shall see, *this* important (:*very restrictive*(!), otherwise) *hypothesis* will finally be *used, only*(!) *in connection with* (4.5)(!), as that was the case in the foregoing, as well: Thus,

(4.22) *no "global use/presence" of* the *euclidean or even locally euclidean space is made*, at all!

This *important fact*, indeed, ensures actually the associated method, as it concerns, at least, its *differential-geometric nature*, its *potential versatility*.

Now, following Sorkin (1991), one chooses the *locally finite open coverings* of X (recall that the latter space is here also *paracompact Hausdorff*, see e.g. (4.12) in the preceding), while one further considers on the *set* X the *topology generated by* such *locally finite open coverings of* X, as above. In this connection we also recall, for occasional use, in relation with ADG, that;

(4.23) the *local frames* of a given *vector sheaf* on a *paracompact* (Hausdorff) space X constitute a *cofinal subset of* the *locally finite open coverings* of X.

See [VS: Chapt. IV; p. 325, (8.42), along with Chapt. II: p. 127; (4.9)]. □

Now, the previous topological spaces, that are associated with locally finite open coverings of X, are further endowed, à la Sorkin (loc. cit.), with appropriate partial orders, becoming thus *"posets,"* alias, *"fintoposets,"* in the terminology of Raptis (2000) (see also Mallios and Raptis (2001)). On the other hand, these toposets are further suitably associated with certain finite-dimensional associative (non-abelian) linear $\mathbb{C}$-algebras, the so-called *"incidence Rota algebras"* (loc. cit.). The same algebras are further sheafified, the resulting sheaves leading finally to appropriate *"differential triads,"* in the sense, of course, that this notion is used by ADG (see [VS: Vol. II], along with Mallios and Raptis (2001, 2002)). Here again, as it also was the case in our previous example in Subsection 4.(a) above, it is of a crucial significance the

(4.24) *possibility of using* the item connected with what we have called in *the* preceding, *"newtonian spark"* (cf. thus (4.5)).

As it was pointed out therein, the latter issue is the *"source"* of the *"differential mechanism,"* that one is supplied with, yet within the present context too,

(4.25) without employing, in effect, the euclidean, or even locally euclidean nature of the origin of that particular *"spark,"* in the way, at least, we are used to do it in the classical theory, thus far!

However, for the technical details thereof, we refer to the relevant work of Mallios and Raptis (2003), along with that one of the same authors in Mallios and Raptis (2005). We have thus herewith still another realization of the fact, being, in point of fact, a *fundamental moral of* ADG (see also Mallios (2004)), that;

(4.26) when we try to apply (*differential*) *geometrical methods*, more so *in the quantum deep*, it seems *more natural* to *apply an analytic* (:algebraic) *way* (with symbols –recall here, for instance, *"Feynman diagrams"*– viz. a *"Leibnizian"* manner of looking at the things, in focus), *not that one of the standard theory* (:*"spatial-newtonian"*).

Yet, what actually leads to the same thing,

(4.27) it is quite natural to try to *concoct*, at each particular case, under consideration, *the appropriate "differential geometric"-machinery* (viz. *"differential triad"*), to cope with the problem at issue.

In toto, we could also mention herewith, a *basic moral of* ADG, in what actually concerns *Quantum Field Theory*. That is,

(4.28) we should *not relate* any (quantum) *field theory with* the existence of an ad hoc given *"continuum"* (:"space-time manifold", whatsoever); this, of course, to the extent, at least, that we wish to apply therein (classical) differential geometry (CDG), since, in that context, the preponderant and really instrumental issue is, in effect, *the* relevant (differential-geometric) *technique* and *not*(!) *the underlying space*.

So, in other words, it is important to afford, in that context, a *"differential-geometric" machinery*, irrespective of the way the latter might have been displayed (cf., for instance, the preceding two examples), while, in any case, *this particular* *way, "spatial," or not* (loc.cit.), *should not intervene in the whole process*, this being especially significant, when referred to the *quantum régime* (see also the relevant comments already in (1.19) in the preceding).

Indeed, in this regard, we can still remark that,

(4.29) as it concerns the *"infinitely small"* (Feynman), *the* (differential) *"geometry,"* in the way, at least, that we use to look at it (viz. in the *"newtonian-cartesian"* one), *is no more valid*(!), since the same –namely, the *"geometry"* becomes –in point of fact, appears to us –in that deep, *more "physical"* (!), as it always is, for that matter, viz. *"relational"* (:algebraic-analytic)!

Exactly at this point, we might also recall the quite relevant remarks here of Finkelstein (1972, p. 155), in that (emphasis below is ours);

(4.30) *"Physics was dominated by the Cartesian epistemology untill the quantum theory."*

Relate the above with our previous considerations in Scholium 2.1 in the preceding. Yet, as a further illumination of the point of view of the whole *formalism of* ADG, we have to point out/clarify, herewith, once more, two fundamental issues of the aforesaid perspective, that also provide a potential outstanding application of the above formalism to ever present problems, thus far, of *quantum gravity*. That is, we have to note, in that context, that:

i) One can employ ADG, *as a* (differential) *"geometry,"* in the classical sense of the latter term, *even in the quantum deep*(!), provided, of course we accept the following correspondence/*"identification"* (:axiomatic),

$$\textit{fields} \longleftrightarrow \textit{vector sheaves}, \tag{4.31}$$

that is, in other words what we have already called elsewhere *"Selesnick's correspondence"* (see, for instance, Mallios (2005, Chapt. II), for a detailed account of this subject matter).

ii) *The same "geometry,"* as above (viz. always, within the framework of ADG), *can still be construed, as* a *"dynamical variable,"* as well (see (2.6), in conjunction with (2.13), as well as, with (2.12)).

On the other hand, another *technical issue*, that should also be pointed out in this regard, is that, *it, very likely, seems* that;

there is no actually *need* to

(4.32.1) *"quantize analysis,"*

as it concerns, in particular, its *topological-linear character* (this being the *source of the Calculus*), since the *inherent/deeper nature* of the same (:of the *"analysis"*), namely, the *"algebraic,"* or even *the*, so to say, *"leibnizian" one, is already*, viz., by its very definition, *"quantized"*!

Yet, by further commenting on our last claim, as above, we still note that;

(4.33) *there is no*, in effect, according to the same definitions, any *"infinite" in* (pure) *algebra*!

So *it is*, therefore, *in "geometry"/topology* (viz. in the so-called, *"infinite"*(!), a consequence, in fact, of the latter perspective), that *we are*, actually, *entangled*, when confronting with the *quantum deep* (:"small distances"). Consequently, our systematic endeavor, up to this day, in one way or another, to succeed in getting an appropriate *"algebraization"* of the whole scenario!

5. Scholium (:more on the **"newtonian spark"**).– We usually *curve a linear structure*, by *"localizing"* it (manifolds); in point of fact, this is a quite *general device*, referring, irrespective of the dimension (finite or infinite), to the (*topological*) *vector space-model* of our (cartesian) *"geometry."* In the case of Analysis, an extraordinary issue, in that context, is that the classical Calculus that traditionally was hospitalized in (even, emanated from) topological vector space-structures (:euclidean spaces) still survived after this transport, a sine qua non, of course, of the justification, for that matter, of the previous movement, suggested, indeed, by particular important applications. Notwithstanding, a *fundamental moral* of the whole issue *of* ADG is that;

the real *corner-stone* of the previous total enterprize is, in effect, what we have already called in the foregoing, the *"newtonian spark,"* in that
(5.1) context, a fact that might also be paralleled with the famous *archimedean demand*, for a pedestal (:"Δός μοί πᾶ στῶ καί τάν γᾶν κινάσω— *"give me somewhere to stand and I shall move the earth"*).

That is, in other words, *following* now *Leibniz*, in what actually concerns (classical) differential geometry (CDG), what one virtually needs is to provide (according to ADG) the appropriate, concerning the particular problem, at issue, *"differential-geometric mechanism"* (!).

Furthermore, what is here of a particular significance, having also important potential applications (even, very likely(!), in *quantum gravity* too), is that:

(5.2) *the* aforesaid *"differential-geometric mechanism,"* in the sense of ADG, *does not* actually *depend*, at all(!), *on any space*, as it was the case, so far, for the classical theory (CDG), the same mechanism *referred* now *directly to the* ("geometric") *objects*, that live on the *"space."*

Indeed, the latter issue in the above remarks, as in (5.2), is, most likely, what already Leibniz, at his time, was looking for! (See, for instance, Bourbaki et al. (1975, Chapt. I; Note historique, p. 161, ft. 1), or even Mallios (2004, (2.1), along with comments following it)).

Thus, by further commenting on (5.2), we can still say, based also on our previous considerations in Mallios (2006a), that;

(5.3) What one actually perceives appears to be the *"sheafification"* of a *"local aspect/information"* pertaining to the particular subject matter in focus. Besides,

(5.3.1) the way we get a *"local information,"* may, in principle, be *entirely different, in character*, from the *mechanism* (:inherent law–*"physical"*/*relational procedure*), *which governs* (hence, the manner too, we should essentially employ the aforesaid *"sheafification,"* viz. the *global aspect of*) *that local information*.

The above explains too what one essentially encounters, in connection with what we have called in the preceding *"newtonian spark."*

Now, the replacement of a *"field"*

$$(\mathcal{E}, D) \tag{5.4}$$

(cf. (2.15)), by its corresponding *"Heisenberg* (:"matrix") *picture,"* viz. by the *"field"*

$$(\mathcal{E}nd\mathcal{E}, D_{\mathcal{E}nd\mathcal{E}}), \tag{5.5}$$

(see Mallios (2004, (9.20)), along with Mallios (2005, Vol. II, Chapt. II; (5.8), (5.11))), hence, via its *"principal sheaf"* version,

$$(\mathcal{A}ut\mathcal{E}, D_{\mathcal{E}nd\mathcal{E}}|_{\mathcal{A}ut\mathcal{E}}), \tag{5.6}$$

as well, may still be viewed, as being in accord with the "impossibility of having a *"relativistic quantum field,"* defined at a point"(!); see, for instance, Bogolubov *et al.* (1975, p. 282, §10.4, p. 283, Theorem (Wightman) 10.6).

On the other hand, (5.3.1), as above, might also be construed, as another effectuation of the classical *"local commutativity,"* or *"microscopic causality" "microcausality"* yet, *"principle of relativistic microcausality,"* or even *"Einstein's locality."*

On the other hand, by further meditating, a bit more, on our previous scholium in (5.3.1), we can actually reformulate it, by remarking in particular that:

(5.7) *the deeper* (algebraic) *mechanism* that might be *inherent in* (:esoteric *to*) *a* given *local information* (alias, a given *local data*), *may* in general *be independent of the way one* has *actually drawn this information* (:*the local data*, concerned).

Yet, in connection with the above remarks in (5.7) and our issue in (4.3), one may recall, in this regard, Wittgenstein's motto (1997, p. 74; 6.54), that;

(5.8) ". . . [one must]. . . *throw away the ladder after he has climbed up it.*"

Now, as a fundamental spinoff of the above, one can still conceive, for instance, the classical (*Machian*) perspective of an

(5.9) *"action at a distance"* (!).

6. Concluding Remarks (the "continuum")– The purpose of this final section is to make clear, once more, that:

(6.1) the notion of the *"continuum,"* as a *"foundational element" is not* actually *the case*, when *physically speaking*, at least (!) (and not only (!), see e.g. (6.5) in the sequel).

Now, the inverted commas put on the word continuum, as above, refer, of course, to the way *we* usually understand that notion in the familiar terminology of the classical theory, where, in point of fact, *we* wish to ascribe to it a physical substance, that is, equivalently, to endow it with a physical meaning. And just hear one has *the crux of the problem*: That is,

(6.2) we are actually influenced by *our mathematical terminology–conception*, in what virtually concerns the word *"continuum"*, i.e., the *"cartesian,"* in point of fact, perspective of the so-called *"space-time."*

Thus, in other words, we make the following *identifications*:

(6.3) *"physical space"* $\longleftrightarrow$ *mathematical "space"* / *"continuum,"* viz. some $\mathbb{R}^n$, as a (finite dimensional) *topological vector space*.

However, it is exactly *the above identifications*, that *is* really *the source of the problems*: Indeed, as we have already remarked in other places (cf., for instance, Mallios (2004, (1.4), or even (3.1))),

(6.4) *"physical space"* is *what* virtually *constitutes it*, that is, in other words, what we may call, à la Leibniz, the *"geometrical objects"* themselves, that make up, what in effect, we perceive, as *"space," in the large*, as well as, *in the small*.

Therefore, in that respect, the substance of the *"physical space,"* as above, is thus *discrete*/*granular*, hence not at all corresponding to something *"continuous,"* viz. *not-discrete*, when physically/conceptually speaking. On the other hand, when *mathematically speaking*, a *set* is already, by its very definition, being thus *"point-wise determined,"* absolutely *"discrete,"* in character!

Thus, by referring to *the mathematical notion of the "continuum," as an* $\mathbb{R}^n$, $n \in \mathbb{N}$,

(6.5) we note that the so-called *"continuum" is*, technically speaking, viz. as a *mathematical term, our* own *definition* of an $\mathbb{R}^n$ ($n \in \mathbb{N}$), as already said, viewed herewith not just, as a *discrete set*, as it actually is, for that matter, but now, as a *topological* (*vector*) *space*, this particular (mathematical) *"structure"* on (the set) $\mathbb{R}^n$ being also the *source of the* (*newtonian*) *Calculus*!

In this connection, we are thus influenced, by our own *mathematical experience* of the concept of the *"continuum,"* in the way we defined it, as above, that is, as a particular *finite dimensional* (*Hausdorff*) *topological vector space*, a point of view that we also attribute, in turn, to what we actually perceive, as a *"physical space,"* this being further construed, as another "continuum," this time, however, as a physical one (!), based rather on a *"dynamically/kinematiccaly"* ascribed description of the (physical) world; alas, something here *in complete conflict with* our actual (:experimental) experience, as it virtually concerns, at least, *the quantum régime* (see also, for instance, (4.30) in the preceding).

Now, in this context, the previous items;

(6.6) *"dynamical-kinematical description"* of the (physical) world, *differential equations-theoretic point of view, Calculus*, and *"space-time continuum"* are, in effect, *intimately related* and, in point of fact, *tautosemous*, in substance.

Strictly speaking, as a matter of fact,

(6.7) *Calculus* is the source/cause of the first two items, as above, while, in turn, the same (Calculus) is *the spin-off*, as already said, *of the newtonian-cartesian*, so far, *definition of the "space,"* that is, of the so-called *"geometrical" perception* of it, yet, *the outcome of* the same *"space-time continuum"* $\equiv \mathbb{R}^n$ (cf. (6.5)).

On the other hand, the above *differential part of the Calculus*, viz. *"differentiation of functions,"* in principle, presupposes *"good*(– (:smooth) differentiable)–*functions,"* something that essentially depends on the *"local behavior"* of the functions concerned; hence, a fact that directly refers to the *local nature of the domain of definition of the* same *functions*, that is, to the *local structure* (:*"geometry"*) of the *"euclidean space,"* $\mathbb{R}^n$, itself. Consequently,

(6.8) we are actually *entangled with the way* the *differential calculus* (:*"differentiation,"* as a mechanism) is supplied (cf. (6.5)), therefore, the *type of the "differentiable" functions* that are *thereby involved*, or, in other words, that are *"locally" defined* on that particular *"space"* (*extremely important*, as well as, *effective* (!), anyhow, concerning the classical theory).

However, the *applications of* the same *differential calculus*, as above, *in* the domain of (classical) *differential geometry*, as a means of study (:working instrument) in that particular discipline, namely, that what we have already considered in the preceding, a *"differential-geometric machinery,"* yet, in other words, a *"geometrical calculus,"* à la Leibniz (see 1.17)), refers, in point of fact, to the very *"geometrical objects"* (Leibniz, loc. cit.), the same being actually (Leibniz, ibid., Riemann, see (Mallios, 2004, (1.3))) *independent of any "space,"* in the sense, at least, of (6.5), as above!

On the other hand, it is reasonable to think that,

(6.9) the very character/substance of what we may call

(6.9.1) *"physical space"* (see also (1.1)) *is*, in principle, *the same*, both *in the large, as well as, in the small*.

We are thus led to a dissonance, by applying our usual *classical representation of the physical space* (in the large), *as an* $\mathbb{R}^n$, irrespective, of course, of the tremendous success, thus far, of the latter perspective, when *realizing*, on the other hand (see also, for instance, (2.2), as well as, (4.30) in the preceding), *that the same* (physical) *"space" is* virtually *quite different from what we are confronted with*, when looking *at the quantum régime*, as it concerns the aforesaid classical perspective; see also Mallios (2004: (8.10), along with (8.11)).

Consequently, the appeared inconveniences (:*"singularities"*), regarding, of course, *applications of classical differential geometry*, as a means of study, in that context, of the *physical space/geometry" in the small*, that is, to say, *physical laws/"fields"* (see also loc. cit., (3.21.1)) *at the "quantum resolution."*

Thus, the *"physical space,"* as a whole, yet, according to recent advances in theoretical physics, concerning, in particular, the *quantum deep*, does not seem to be the usual *"space-time" manifold*, in the sense of the *classical differential geometry*–theory of *smooth* (:$\mathcal{C}^{\infty}$-)*manifolds*. Indeed, it appears that we have therein,

(6.10) something *foamy, very singular*, or even something like what we may call, a *"singularity manifold,"* to refer, in that respect, to a rather recent utterance of R. Penrose, pertaining to a *"true theory of quantumgravity,"* by replacing the *"present concept of spacetime at a singularity."* (See also, for instance, Mallios (2004, (10.8), along with the subsequent discussion therein]).

Thus, concerning the *"infinitely small,"* or else *"quantum resolution,"*

(6.11) the *"geometry,"* in the way we usually look at it (viz. in the *"newtonian-cartesian"* manner), is *no more valid* (!), in that context, since, at that deep, the same *becomes* thus, even to our senses (!), more *"physical,"* as, in point of fact, it always is, for that matter (see also (1.1) in the preceding, along with (6.9) above), viz. *"relational"* (:algebraic-analytic); the latter aspect is *still a fundamental issue*, in effect, of our experience, thus far, *from* ADG, as well!

Indeed, as already advocated in several places in the preceding, working within the context of ADG, we are able to look at fundamental concepts of physics, as, for instance, particles/fields (see also Mallios (2006b, (3.2))) etc, without being compelled to stick to such *"technical" notions*, as e.g. *"space-time"* (!). In this regard, see also, for instance, (1.8) in the preceding, yet, loc. cit. (1.6).

Yet, in this connection, we are thus very likely, led to conclude that the entanglement of the *"manifold"* perspective in nowadays physics, especially, *in the quantum domain*, is to be attributed, in effect, to the *relation of the former with the notion of "differential"*; indeed, the latter is the main function, that is actually applied in our relevant rationale, in that context, while finally, as an upshot of the classical theory (:differential geometry), we are usually of the opinion that *the* same *manifold* concept *is thus the unique*(!) *source of* the notion of a *"derivative,"* covariant or not! *Therefore, the significance,* hereupon *of* the new proposal, as this is, provided, by *ADG*: That is, once more, we realize that

(6.12) *to have a connection, we do not* actually *need a "manifold,"* even if we momentarily borrow from such a concept the *"infinitesimal* (:*"newtonian"*) *spark*! The latter constitutes, in point of fact, the *central moral* of the entire study *of ADG*.

Thus, by looking at the *standard question* (cf., for instance, Sharpe (1997, p. 2, ft. 2)),

(6.13) *"what geometry on a manifold supports physics?,"*

we can combine it now with the *fundamental moral of ADG* (see e.g. (6.12), as above, along with Mallios (2004, (1.2), (3.21.2)), yet, Mallios (1998b)), that, in point of fact,

(6.14) *differential geometry* means, in effect, *connection*.

Consequently, blending the previous two aspects, as in (6.13) and (6.14), we are thus led to a *response to* (6.13), in the sense of affording a *pertinent choice of* "$\mathcal{A}$," more precisely speaking, the associated with it *"differential triad"* and the concomitant *"differential-geometric mechanism,"* à la ADG, suitable to the particular problem at issue; see thus, for instance, Subsections 4. (a), 4. (b) in the preceding, along with Mallios (1998b), p. 174; concluding remarks). In this regard, see also the latest relevant account in Mallios and Raptis (2005).

> *"Le caractère propre des méthodes . . . consiste dans l'emploi d'un petit nombre de principes généraux . . . ; et les conséquences sont d'autant plus étendues que les principes eux-mêmes ont plus de généralité."* — G. Darboux

> *". . . unification of interactions is achieved through unification of ideas."* — V. Knizhnik

6.(a). ADG vis-à-vis a Unified Field Theory.– The quite ambitious(!) title of the present Subsection is rooted, in point of fact, on our previous *comments in* (6.9.1) and on the very essence of the point of view of the same *Abstract Differential Geometry* (ADG), as the latter can be applied, in that context, in conjunction, for instance, with *Rosinger's theory of "generalized functions"*: The technical part of the aforesaid scheme, hinted at herewith, has been already expounded in Mallios and Rosinger (1999,2001); in this connection, see also our previous discussion in Subsection 4.(a) in the foregoing, along with Mallios (2006b, (5.20), (5.21)), as well as, (Mallios, 2004, Sections 6, 8; see, in particular, (8.8) therein, or even (8.11), yet, Section 10). Moreover, cf. also Mallios (2005, Vol. II, Chapt. IV; Sections 5 and 6, along with Section 10 therein, see e.g. (10.29)).

Now, as already said, the preceding just hint at a *potential confrontation* with the second issue in the title of this Subsection, through the machinery of ADG, that is, to say, in terms of the *techniques of the classical differential geometry*, being, however, *freed now from* the ever disturbing/pestilential *"singularities,"* and the like, of the classical approach to the problem at issue. Of course, this is due here, as already explained, throughout the preceding, to the *absence*, according to ADG, *of any supporting "space," that would also exclusively supply* (:generate) *the "differential-geometric" machinery employed*, in that context (see the relevant citations, as before), a situation inherent, in effect, in the classical theory (CDG; cf. the previous Section 5, along with the concluding remarks above, preceding the present Subsection).

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